• Laurent[1] – Dirichlet pb in varying domains

I) Introduction :

u_Ω^f will be the unique solution of

$$\begin{cases} -\Delta u = f \\ u \in H_0^1(\Omega) \end{cases} \qquad \Omega \subseteq \mathbb{R}^N \text{ open, bdd}$$

$$\updownarrow$$

$$\min_{H_0^1(\Omega)} \left[\frac{1}{2} \int_\Omega |\nabla v|^2 - \langle f, v \rangle_{H^{-1} \times H_0^1} \right] = \frac{1}{2} \int_\Omega |\nabla u|^2 - \langle f, u \rangle$$

• convention : when $\Omega = \emptyset \rightsquigarrow u_\emptyset^f = 0$.

Kind of a maximum principle. $\rightsquigarrow$ • $f \longmapsto u_\Omega^f$ is non-decreasing (in the sense of distributions)

• $f \geq 0 \rightsquigarrow \Omega \longmapsto u_\Omega^f$ is non-decreasing

$\to$ Fix $D \subseteq \mathbb{R}^N$ open, bdd, vary $\Omega \subseteq D$, fix $f \in H^{-1}(D)$.
[$\Rightarrow f \in H^{-1}(\Omega)$ $\forall \Omega \subseteq D$].

• question : $\Omega_n \longrightarrow \Omega \Rightarrow u_{\Omega_n}^f \overset{?}{\longrightarrow} u_\Omega^f$

$\to$ **NO**, without any further assumption on $(\Omega_n)_n$.

• Notice: if we have a positive answer for $f \equiv 1$, then the same holds true for every f.

• Prop: $$\int_D |\nabla u_\Omega^f|^2 = \langle f, u_\Omega^f \rangle \Rightarrow \|u_\Omega^f\|_{H_0^1(D)} \leq c\|f\|_{H^{-1}(D)}$$

• Cor: we can extract from $u_{\Omega_m}^f \longrightarrow u^*$ in w-$H_0^1(D)$.

If $u^* = u_\Omega^f$ for some $\Omega \subset D$, then
$$u_{\Omega_n}^f \longrightarrow u^* \text{ in } H_0^1(D).$$

[by **If** we have convergence of the norms of $|\nabla u_{\Omega_n}^f|$]

• Note: if $\exists\, \Omega \subset D$ s.t. $\forall K \subset\subset \Omega$ $\exists\, m_0 \in \mathbb{N}$
$\forall m \geq m_0$ $K \subset \Omega_m$,] (*)
then $\exists\, u^* \in H_0^1(D)$ s.t.
$$\int_\Omega \nabla u^* \cdot \nabla v = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega).$$

$\boxed{\text{Pb: here we are \underline{not} ensured that } u^* \in H_0^1(\Omega)}$ **!**

$\rightarrow$ If yes, $u_{\Omega_m}^f \rightarrow u_\Omega^f$ s-$H_0^1(D)$

II) Independence of f

- Thm: (Sverak, '92)

$$u_{\Omega_m}^{1} \xrightarrow{L^2(D)} u_{\Omega}^{1} \Rightarrow u_{\Omega_m}^{f} \xrightarrow{L^2(D)} u_{\Omega}^{f} \quad \forall f \in H^{-1}(D)$$

Proof:

- step 1: $f \in L^\infty(D)$: $\|f\|_{L^\infty} \leq M$, by max. principle & linearity, we get:

$$-M u_m^1 \leq u_m^f \leq M u_m^1$$

$$\xrightarrow{w-H_0^1(D)} u^*$$

→ we want to show: $u^* = u_\Omega^f$.

We have: $-M u_\Omega^1 \leq u^* \leq M u_\Omega^1$

$$\Downarrow$$

$$u^* \in H_0^1(\Omega)$$

Now pick: $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$,

since $u_\Omega^1 > 0$ on Ω [by the strong max princ.]

$\Rightarrow$ $\varphi \leq p u_\Omega^1$ [$p>0$, ok because φ has compact support]

Let $\varphi_m := \inf\{\varphi, p u_m^1\} \in H_0^1(\Omega_m)$

Then $\varphi_m \longrightarrow \varphi$ s-$H_0^1(D)$.

We have:
$$\int_D \nabla u_m^f \cdot \nabla \varphi_m = \int_D f \varphi_m$$
$$\downarrow \qquad\qquad \downarrow$$
$$\int_D \nabla u^* \cdot \nabla \varphi = \int_D f \varphi$$

By splitting any $\varphi = \varphi^+ - \varphi^-$ $\rightsquigarrow$ $u^* = u_\Omega^f$.

- Step 2: (not trivial) we conclude by a density argument.

- **Notice**: true also if $f_m \to f$ $H^{-1}(D)$.

III) $\Omega_m \to \Omega$: different scenarii:

- Thm 1:

If $\Omega_m \subseteq \Omega_{m+1}$, $\Omega := \bigcup_m \Omega_m$, then:
$$u_m \xrightarrow{s-H_0^1(D)} u$$
More generally, if $\Omega_m \subseteq \Omega$, $\Omega_m \xrightarrow{\text{Hausdorff}} \Omega$,
$$u_m \xrightarrow{s-H_0^1(D)} u.$$

Proof:

- By Hausdorff convergence, we have the validity of hyp. (*).
- We need to prove: $u^* \in H_0^1(\Omega)$. ← boundary condition!

notice that: $u_m \in H_0^1(\Omega_m)$

$\Rightarrow \exists\, \varphi_m \in C_c^\infty(\Omega_m)$ s.t. $\|\varphi_m - u_m\|_{H_0^1(D)} \leq \frac{1}{m}$

$$\varphi_m \xrightarrow{H_0^1(D)} u^*$$

but $\Omega_m \subseteq \Omega \Rightarrow \varphi_m \in C_c^\infty(\Omega)$

$\Rightarrow u^* \in H_0^1(\Omega)$.

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→ if Ω_m goes to Ω from the inside, everything works!

- Corollary:

If $\Omega_m \xrightarrow{H} \Omega$, we can say that:

$$u^f_\Omega(x) \leq \liminf_m u^f_{\Omega_m}(x) \quad \text{a.e. in } D.$$

• **Thm 2**: [dimension 1]

Let $D \subseteq \mathbb{R}$ is an interval, and $\Omega_m \xrightarrow{H} \Omega \subset D$, then

$$\begin{cases} u_m \longrightarrow u & s\text{-}H_0^1(D) \\ u \in H_0^1(\Omega) \end{cases}$$

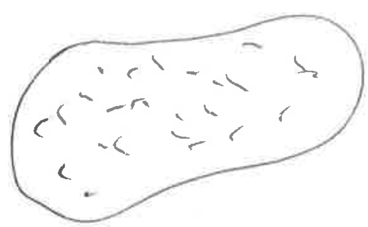
Proof:

$H_0^1(D) \hookrightarrow C^{0,\alpha}(D)$; so if $x \in D\setminus\Omega$, then $\exists x_m \in D\setminus\Omega_m, x_m \to x$; but $u_m(x_m)=0$

$u(x)=0$

(2)

• Example:



D

let $(x_m)_m \subset D$ dense

$\Gamma_m := \{x_0, \dots, x_m\}$

$\Omega_m := D \setminus \Gamma_m$.

Then $\Omega_m \xrightarrow{H} \emptyset$.

But, $u^f_{\Omega_m} \not\to 0$; because the H^1 norm does **not** see Γ_m!

$$\boxed{u^f_{\Omega_m} = u^f_D \quad \forall m}$$

$$\boxed{H_0^1(\Omega_m) = H_0^1(D)}$$

this is a matter of capacity!

- Example: (Murat–Tartar)

Let $D = (0,1)^2$, $x_{i,j} := \left(\frac{i}{n}, \frac{j}{n}\right)$

$$\Omega_n := D \setminus \bigcup_{i,j} B(x_{i,j}, r_n)$$

- case 1: if $\frac{\log r_n}{n^2} \to -\infty$ then $u^* = u_D$ [if small enough]

- case 2: if $\frac{\log r_n}{n^2} \to 0$, then $u^* = 0$

- case 3: if $\frac{\log r_n}{n^2} \to -d < 0$, then

  u^* solves: $\quad -\Delta u^* + \frac{2\pi}{d} u^* = f$

  new term!

  [coming from the excess of Ω_n over Ω

$\to$ what changes is: $\boxed{\mathrm{cap}(\Omega_n; \Omega)}$

• **Def:** $\operatorname{cap}(E) := \inf \left\{ \|v\|_{H^1(\mathbb{R}^N)} \, , \, v \geq 1 \text{ a.e. on a nghbh of } E \right\}$.

• Lemma:

Let $(\Omega_n) \subseteq \mathbb{R}^N$, $(u_n)_n \subseteq H_0^1(\Omega_n)$

$u_n \xrightarrow{w\text{-}H_0^1(\mathbb{R}^N)} u^*$; let $\tilde{\Omega}_p := \bigcup_{n \geq p} \Omega_n$,

$E := \bigcap_p \tilde{\Omega}_p$.

Then $\boxed{\tilde{u}^* = 0}$ [the quasi-continuous representant of u^*] outside E, cap-a.e.

Moreover, if $\exists A$ s.t. $\operatorname{cap}(\Omega_n \setminus A) \to 0$, then $\tilde{u}^* = 0$ cap.-a.e. outside A.

• Thm:

If $\Omega_n \xrightarrow{H} \Omega$ and $\operatorname{cap}(\Omega_n \setminus \Omega) \to 0$, then

$$\begin{cases} u^f_{\Omega_n} \to u^f_\Omega \quad s\text{-}H_0^1(D), \\ u^f_\Omega \in H_0^1(\Omega). \end{cases}$$

Proof:

by the lemma, with $A = \Omega$.

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Proof (of the lemma)

- By Mazur's lemma $\exists (v_p)_p$ convex combination of u_m, $m \geq p$ with $v_p \xrightarrow{s-H_0^1} u^*$.

  Each $\tilde{u}_m = 0$ cap.-a.e. outside Ω_m

  So $\tilde{v}_p = 0$ cap.-a.e. outside $\tilde{\Omega}_p$

  Up to a subsequence, $\tilde{v}_p \longrightarrow \tilde{u}^*$ a.e.

  $\Rightarrow \tilde{u}^* = 0$ cap.-a.e. outside E.

- if $\mathrm{cap}(\Omega_m \setminus A) \leq 2^{-m} \Rightarrow \mathrm{cap}(\tilde{\Omega}_p \setminus A) \leq 2^{-p+1}$

  $\Rightarrow \mathrm{cap}(E \setminus A) = 0$.

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